FOURIER TRANSFORM AND HEAT FLOW

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We can extend the Fourier series solution for the heat flow problem to an infinite rod by using the Fourier transform. The solution satisfies the heat equation

$$\frac{\partial T(x,t)}{\partial t}=\frac{\partial^2 T(x,t)}{\partial x^2} \tag{1}$$

with the initial condition

$$T(x,0)=f(x) \tag{2}$$

The Fourier transform of $f(x)$ is

$$G(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)\,e^{-ix\omega}dx \tag{3}$$

with the inverse transform

$$f(x)=\int_{-\infty}^{\infty}G(\omega)\,e^{i\omega x}d\omega \tag{4}$$

A solution to 1 is given by

$$T(x,t)=\int_{-\infty}^{\infty}G(\omega)\,e^{i\omega x}e^{-\omega^2 t}d\omega \tag{5}$$

To verify this, we have, assuming we can differentiate under the integral sign

$$\begin{aligned}
\frac{\partial T(x,t)}{\partial t} &= \int_{-\infty}^{\infty}G(\omega)\,e^{i\omega x}\left(-\omega^2\right)e^{-\omega^2 t}d\omega && (6)\\
\frac{\partial^2 T(x,t)}{\partial x^2} &= \int_{-\infty}^{\infty}G(\omega)\,(i\omega)^2\,e^{i\omega x}e^{-\omega^2 t}d\omega && (7)\\
&= \int_{-\infty}^{\infty}G(\omega)\,e^{i\omega x}\left(-\omega^2\right)e^{-\omega^2 t}d\omega && (8)\\
&= \frac{\partial T(x,t)}{\partial t} && (9)
\end{aligned}$$

The initial condition 2 is

$$T(x,0)=\int_{-\infty}^{\infty}G(\omega)\,e^{i\omega x}d\omega=f(x) \tag{10}$$

We can eliminate $G(\omega)$ from 5 by using 3.

$$T(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}d\xi\ f(\xi)\,e^{-i\xi\omega}e^{i\omega x}e^{-\omega^{2}t} \tag{11}$$

where we've introduced the dummy variable ξ for the second integration. We can reverse the order of integration to get

$$T(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi\ f(\xi)\int_{-\infty}^{\infty}d\omega e^{-\omega^{2}t+i\omega(x-\xi)} \tag{12}$$

The ω integral can be evaluated explicitly by completing the square in the exponent. The exponent is

$$-\omega^{2}t+i\omega(x-\xi)=-t\left(\omega^{2}-\frac{i\omega(x-\xi)}{t}\right) \tag{13}$$

$$=-t\left(\omega-\frac{i(x-\xi)}{2t}\right)^{2}-t\frac{(x-\xi)^{2}}{4t^{2}} \tag{14}$$

$$=-t\left(\omega-\frac{i(x-\xi)}{2t}\right)^{2}-\frac{(x-\xi)^{2}}{4t} \tag{15}$$

The last term doesn't involve ω so can be pulled outside the ω integral, giving

$$T(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi\ f(\xi)\,e^{-(x-\xi)^{2}/4t}\int_{-\infty}^{\infty}e^{-t(\omega-i(x-\xi)/2t)^{2}}d\omega \tag{16}$$

The last integral can be evaluated using the substitution

$$u=\sqrt{t}\,(\omega-i(x-\xi)/2t) \tag{17}$$

$$du=\sqrt{t}\ d\omega \tag{18}$$

$$\omega=-\infty\rightarrow u=-\infty \tag{19}$$

$$w=\infty\rightarrow u=\infty \tag{20}$$

The integral is then

$$\int_{-\infty}^{\infty}e^{-t(\omega-i(x-\xi)/2t)^{2}}d\omega=\frac{1}{\sqrt{t}}\int_{-\infty}^{\infty}e^{-u^{2}}du \tag{21}$$

$$=\sqrt{\frac{\pi}{t}} \tag{22}$$

where we used the result for the Gaussian integral

$$\int_{-\infty}^{\infty}e^{-u^{2}}du=\sqrt{\pi} \tag{23}$$

Inserting back into 16 we get

$$T(x,t)=\frac{1}{2\pi}\sqrt{\frac{\pi}{t}}\int_{-\infty}^{\infty}d\xi\ f(\xi)\,e^{-(x-\xi)^{2}/4t} \tag{24}$$

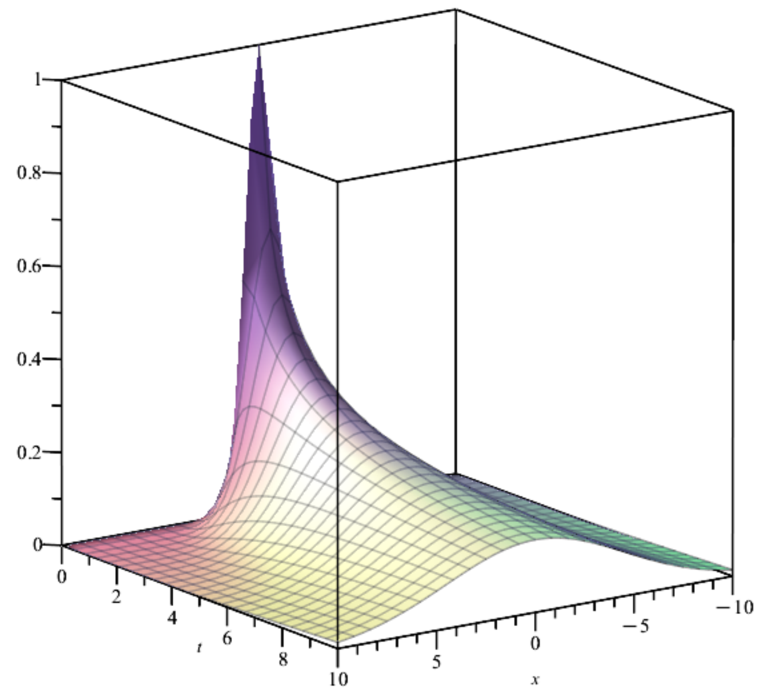

FIGURE 1. Plot of $T(x,t) = \frac{e^{-x^2/(4t+1)}}{\sqrt{4t+1}}$ for $x \in [-10, 10]$ and $t \in [0, 10]$.

$$= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} d\xi \; f(\xi)\, e^{-(x-\xi)^2/4t} \tag{25}$$

As an example, we take the initial condition to be

$$f(\xi) = e^{-\xi^2} \tag{26}$$

This gives

$$T(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} d\xi \; e^{-\xi^2} e^{-(x-\xi)^2/4t} \tag{27}$$

$$= \frac{e^{-x^2/(4t+1)}}{\sqrt{4t+1}} \tag{28}$$

where I used Maple to do the integral. To do it by hand, you could employ the completing the square technique used above.

The plot of $T(x,t)$ is given in Fig. 1. As we might expect, the temperature tends to decrease and spread out along the rod.